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"Color Matching Functions When One Primary Wavelength is Changed: A General Rule and Relation to Least-Power Primaries."

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Abstract

Color-matching functions (cmfs) produced by monochromatic primaries change in an orderly way when the wavelengths of the primaries are changed. When only one of the three wavelengths is varied, the corresponding cmf changes in scale but not in shape. That is, changing one primary changes all three cmfs, but if only the red primary is changed, for example, then the red cmf changes only in scale. A set of primaries will exist such that each cmf has a maximum value of 1, and that peak occurs at the primary wavelength, by a prior theorem. Those are Thornton's prime colors, and if they are the initial primaries, then changing one primary wavelength can only increase the scale of its cmf. Precise prime color sets have been calculated: (603, 538, 446) for the 2-degree observer, and (600, 536, 445) for the 10-degree observer.

Keywords: Color Matching; Color Theory; Prime Colors; Thought Experiment; Animated Graph.

Introduction.

The color matching experiment is a foundation of colorimetry, and a prototype for devices in which primary colors are added, such as color television. When one set of color matching functions (cmf's) is known, they can be transformed to predict the cmf's for any set of primaries. This article concerns a thought experiment with three narrow-band primaries, and what happens when the wavelength of one primary is changed. Such thought experiments, supported by graphical illustrations, may enhance intuition about the connection of primary sets and color-matching functions.

Consider Figure 1. The thin solid lines are cmf's for one experiment in which the CIE's 2° observer matches a test light against primaries of 603, 538, and 446 nm. The thicker dashed lines apply when the primary wavelengths are 650, 538, and 446 nm. In other words, just the red primary changes, shifting from 603 to 650. Since the green cmf must pass through zero at the red and blue primary wavelengths[1], it clearly changes shape when the red primary is shifted. The zero crossings of the red cmf itself are not changed between the two cases, although the function changes in scale. We now show that the red cmf changes only by a scale factor.

One CMF and its Primary Wavelength

Assertion: When one primary wavelength is changed (say the red wavelength only) then the associated (red) color matching function changes only in scale, not in shape.

Thought experiment: Consider a hypothetical color matching experiment, in which the subject views a test field of unit power and adjustable wavelength λ, and sets a matching light in the comparison field. For the comparison field, the experimenter chooses wavelengths μ_1, μ_2, μ_3 for the narrow-band primaries. The subject makes a match by adjusting the amplitudes in the comparison channels, with wavelengths remaining constant. At most settings of λ, one of the primaries is in fact added to the test, and its amplitude is then considered to be a negative number. In a colorimetry laboratory, the primary wavelengths might remain fixed for a long time. We are now concerned with what happens or does not happen when one of the wavelengths, say μ_1, is increased or decreased from its initial value.

For each setting of the independent variable λ, 3 dependent quantities result, the subject's power settings for the primary lights. Thus, 1 unit of λ matches $r(\lambda)$ units of μ_1, plus $g(\lambda)$ units of μ_2, plus $b(\lambda)$ units of μ_3. If they match, then we can predict that the matching lights will have the same tristimulus vector, based on prior data. For convenience, and without loss of generality, we can say that they give equal tristimulus vectors in the CIE's XYZ system:

$$\begin{bmatrix} \bar{x}(\mu_1) & \bar{x}(\mu_2) & \bar{x}(\mu_3) \\ \bar{y}(\mu_1) & \bar{y}(\mu_2) & \bar{y}(\mu_3) \\ \bar{z}(\mu_1) & \bar{z}(\mu_2) & \bar{z}(\mu_3) \end{bmatrix} \begin{bmatrix} r(\lambda) \\ g(\lambda) \\ b(\lambda) \end{bmatrix} = \begin{bmatrix} \bar{x}(\lambda) \\ \bar{y}(\lambda) \\ \bar{z}(\lambda) \end{bmatrix} . \tag{1}$$

On the right-hand side of Eq. (1) is the vector for the test light. On the left, each term of the tristimulus vector for the matching light is found by a sum, computing a tristimulus vector from the three power settings. Since it holds for each λ, if Eq. (1) can be solved, the entire table (or graph) of r, g, b can predicted from a table of $\overline{x}$, $\overline{y}$, $\overline{z}$. Cramer's rule is now applied to solve for $r(\lambda)$:

$$r(\lambda) = \frac{\begin{vmatrix} \overline{x}(\lambda) & \overline{x}(\mu_2) & \overline{x}(\mu_3) \\ \overline{y}(\lambda) & \overline{y}(\mu_2) & \overline{y}(\mu_3) \\ \overline{z}(\lambda) & \overline{z}(\mu_2) & \overline{z}(\mu_3) \end{vmatrix}}{\begin{vmatrix} \overline{x}(\mu_1) & \overline{x}(\mu_2) & \overline{x}(\mu_3) \\ \overline{y}(\mu_1) & \overline{y}(\mu_2) & \overline{y}(\mu_3) \\ \overline{z}(\mu_1) & \overline{z}(\mu_2) & \overline{z}(\mu_3) \end{vmatrix}} \quad . \tag{2}$$

In Eq. (2), the denominator is not a function of λ, so for each setting of μ_1, μ_2, μ_3 , it is a constant that does not affect the shape of $r(\lambda)$. The numerator does not depend on μ_1. Therefore, as μ_1 is varied, the function $r(\lambda)$ may be scaled up and down because of the changing denominator, but that is the only change. The shape of $r(\lambda)$ does not otherwise change, meaning, for example, that the wavelength of its peak does not change. The same principle holds for the other primaries μ_2, μ_3, and their associated functions $g(\lambda)$, $b(\lambda)$.

More General Statement. In the laboratory or in a thought experiment, one needs the test light to be a more or less narrow band centered at wavelength λ , otherwise the meaning of functions r, g, b, graphed versus λ , will be unclear. The same is not true of the primaries. Suppose that the red primary is not a narrow band at μ_1, but has a spectral power density, $p_1(\nu)$, where ν is wavelength. Then in Eq. (1), terms such as $\overline{x}(\mu_1)$ are replaced by inner products, for example,

$$\overline{x}(\mu_1) \rightarrow \langle \overline{x} p_1 \rangle = \sum_{\nu=360}^{830} \overline{x}(\nu) p_1(\nu) \quad . \tag{3}$$

The inner products such as $\langle \overline{x} p_1 \rangle$ carry along into Eq. (2), so the color matching function $r(\lambda)$ varies only by a scale factor as the function $p_1(\nu)$ is changed. To say it more dramatically, the shape of the red cmf $r(\lambda)$ depends only on the functions p_2 and p_3 defining the blue and green primaries, and not on the function p_1 defining the red primary itself. All three primaries can be non-narrow bands, and the same principle holds. In all cases, the functions must satisfy the criterion that the denominator determinant in Eq. (2) is nonzero. The notion of Prime Colors, reviewed below, does depend on narrow primaries.

Broader Context

In general, all three primary wavelengths are parameters that can be arbitrarily set, but that does not mean all primary sets are created equal. Remarkably, the three color matching functions tend to peak at certain preferred wavelengths, which W. Thornton reported as 604, 541, and 447 nm for the 2° observer[2]. These numbers are averages found after locating the peaks for 792 unique combinations of primaries[2]. Thornton's preferred wavelengths, the theorem above, and other facts[1,3,4], come to life if the graph of color matching functions is animated by stepping through primary sets at short time intervals[5]. Such an animation is available on http://www.jimworthey.com/matchingprime.html .

Primary Election. In the race to become the red primary, 603 nm and 650 nm are not equally qualified candidates. The narrow band at 603 nm requires less power at its peak. Simple reasoning says that when wavelength λ of the unit power test light equals μ_1, then unit power of μ_1 and zero power in the other primaries will match the test. By the theorem just proven, if one adjusts only μ_1, holding μ_2, μ_3 constant, the peak will not move left or right. The wavelength of the peak happens to be 603 nm, and if one sets μ_1 = 603, the peak amplitude is found to be unity. Therefore 603 nm is a least-power setting for μ_1, given that μ_2 = 538, μ_3 = 446.

If the set of primaries $\{\mu_1, \mu_2, \mu_3\}$ is optimized so that each of the color matching functions is 1 at its peak, then the peaks will coincide with the primary wavelengths. By Thornton's most refined definition, those wavelengths are the Prime Colors[6]. In 1971, Thornton showed that three wavelength regions were "most effective" in the context of color mixtures and color rendering. Later he introduced the name Prime Colors for these special wavelengths, and defined them in terms of color matching experiments as what we can call the least power primaries.

The least-power property of the prime-color wavelengths shows up in the fact that those wavelengths maximize the determinant in the denominator in Eq. (2)[1] . The peak of $r(\lambda)$ occurs when the derivative of the numerator with respect to $\lambda = 0$, meaning that the numerator is at a maximum. For the peak to be at the primary wavelength, it must occur when numerator = denominator. Therefore, the derivative of the denominator with respect to μ_1 is also zero. The argument extends to μ_2 and μ_3 as well.

By the least power criterion, and using CIE color matching functions that are interpolated and smoothed to 1-nm intervals[8], we have computed the prime color sets in Table 1.

Table 1. Prime color sets (= least power primaries) as defined by Thornton.

	red	**green**	**blue**
1931 2° Observer	603 nm	538	446
1964 10° Observer	600	536	445

JAW first wrote an algorithm based directly on the concept that the least power primaries should coincide with the peaks. A set of primaries is chosen arbitrarily and the related cmf's are found, then the peaks of those cmf's. Those peaks become the primaries in the next iteration; new cmf's are found, and then their peaks. Wavelength is treated as an integer, so when the primary set repeats once, the process ends, typically after 5 iterations. When thousands of starting conditions are tried, a few cases fail to converge, the rest lead to the same result, and all peak amplitudes in the final iteration are 1.0000000, based on double-precision calculation.

Relying on the theorem in Ref. 1, MHB independently wrote an algorithm to find the set of wavelengths which maximize the determinant in the denominator of Eq. (2). The same wavelength sets result. JAW then wrote a different program based on MHB's idea, and again the same wavelength sets were found.

Conclusion

In a notional color matching experiment with narrow-band primaries, if only one primary wavelength is changed, the associated color matching function will change in amplitude, but not in shape. This theorem corroborates Thornton's observation that the peaks of the cmf's tend to stay in narrow wavelength ranges[2]. Numerical values of the Prime Color wavelengths are presented, based on prior theorems.

Acknowledgment. The animation was created first, with assistance from Nicholas J. Worthey. It inspired MHB to discover the theorem.

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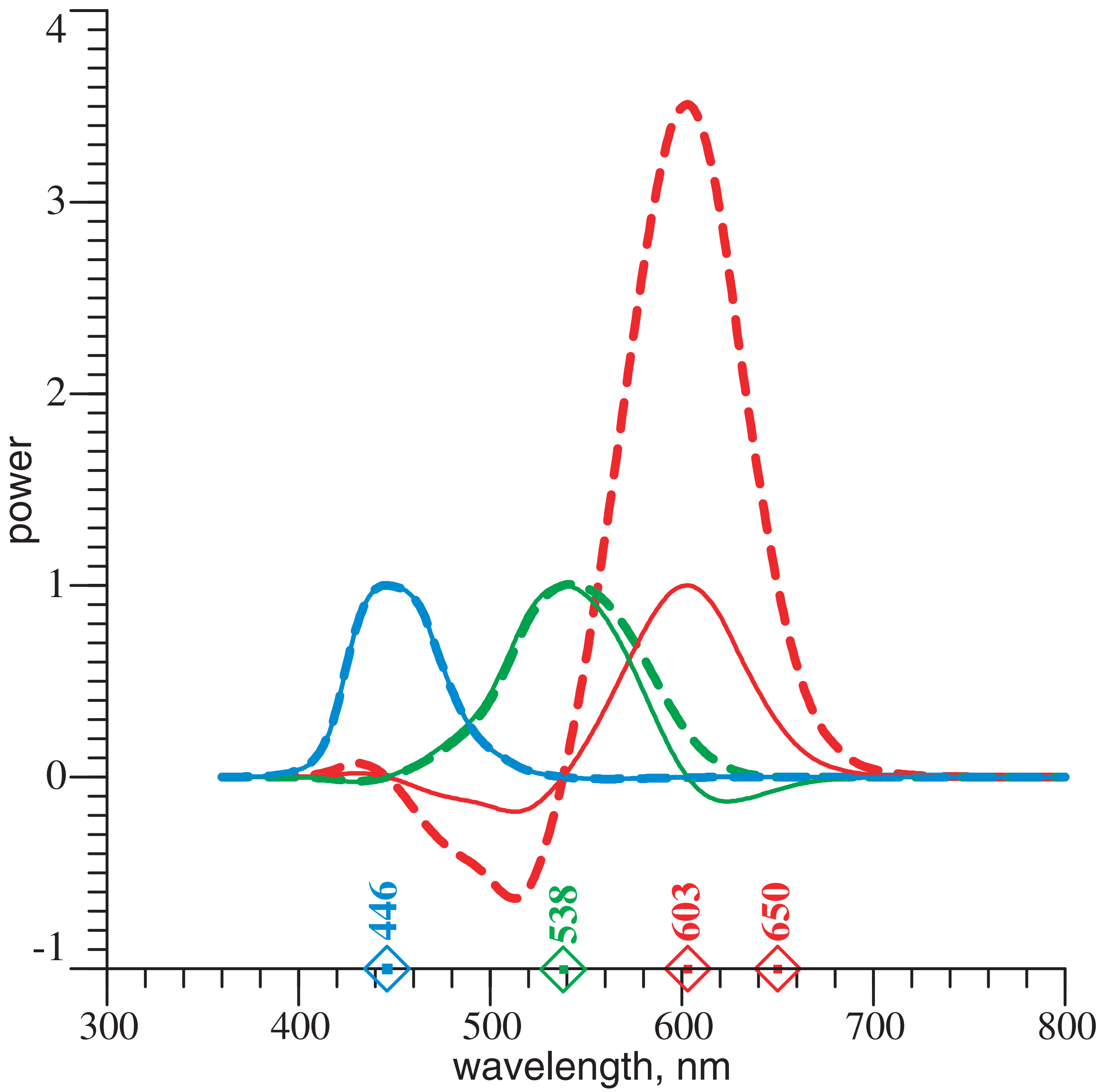

Figure 1. Predicted color matching functions for the CIE's 2° observer. When the observer matches a test light with primaries of 603, 538, and 446 nm, the thin solid functions apply. The thicker dashed lines are the result when the blue and green primaries are unchanged, but the red one is set to 650 nm.