Orthonormal Opponent Functions

$$\Omega = [|\omega_1\rangle, |\omega_2\rangle, |\omega_3\rangle] \qquad (12)$$

1. ω_1 is achromatic sensitivity (whiteness). the function ω_1 is proportional to the usual $\bar{y}$, but normalized. That is $\langle\omega_1|\omega_1\rangle = 1$. ω_1 is a sum of (red cones) + (green cones), with appropriate coefficients. There is no blue input to ω_1.
2. There is also no blue input to ω_2. ω_2 is a difference, (red cones) − (green cones), with coefficients such that it is orthogonal to ω_1.
3. The 3rd function, ω_3 is the most messy. It has inputs from blue, red, and green cones.

Orthonormality

$$\langle\omega_i|\omega_j\rangle = \delta_{ij} \quad , \qquad (13)$$

meaning:

$$\langle\omega_i|\omega_j\rangle = 1 \text{ if } i = j \quad ,$$
$$\langle\omega_i|\omega_j\rangle = 0 \text{ if } i \neq j \quad .$$

Notation

$|f\rangle$ = function f as a column vector.
$\langle f| = |f\rangle^T$ = function f as a row vector.

$|f\rangle$ = “ket”
$\langle f|$ = “bra”
$\langle f|g\rangle$ = “bracket”

Bras and kets distinguish row and column vectors.

By ordinary matrix multiplication, a complete bracket indicates the inner product of two vectors. For example:

$$\langle f|g\rangle = \Sigma f_\lambda g_\lambda \quad . \qquad (14)$$

The sum is taken over the vector length, as a proxy for the working spectrum. The vectors might have length 31 or 471, for example.

On the other side of this sheet is an Overview of Algebra that covers more topics in condensed form.

Overview of Algebra for Vectorial Color

Begin with a set of 3 color matching functions, CMFs: $\{q_1, q_2, q_3\}$. The functions q_j could be cone sensitivities $\{r, g, b\}$, data from a color matching experiment, the CIE basis $\{\bar{x}, \bar{y}, \bar{z}\}$, or something else. Let the 3 functions be written as column vectors, which become the columns of a matrix A:

$$A = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \vdots & \vdots & \vdots \end{bmatrix} \quad . \tag{1}$$

Now let L_1 be the spectrum of any light, also written as a column vector. The Fundamental Metamer of L_1 is the projection of L_1 into the column space of A. The fundamental metamer of L_1 is by definition written as L_1^* . Cohen found that if projection matrix **R** is

$$\mathbf{R} = A[A^{\mathrm{T}}A]^{-1}A^{\mathrm{T}} \quad , \tag{2}$$

then

$$L_1^* = \mathbf{R}L_1 \quad . \tag{3}$$

An alternate wording is that L_1^* is that linear combination of the columns of A which is a least-squares best fit to L_1. **R** is a large square matrix of numerical constants that is easily handled in a computer, but would be cumbersome to print out. It might have dimensions 31×31 or 471×471, for example.

Invariance of R. It can be shown that **R** is invariant to a linear transformation on the columns of A. That is, the columns of A can be cone sensitivities or $\bar{x}, \bar{y}, \bar{z}$, or some other equivalent CMFs, and the result **R** comes out exactly the same.

One linear transformation of CMFs is $\boldsymbol{\Omega}$, the orthonormal basis. That is,

$$\boldsymbol{\Omega} = [|\omega_1\rangle, |\omega_2\rangle, |\omega_3\rangle], \text{ where } \langle\omega_i|\omega_j\rangle = \delta_{ij} \quad . \tag{4}$$

Letting $A = \boldsymbol{\Omega}$ does not change **R**. The orthonormal property can be expressed in matrix form:

$$\Omega'\Omega = I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \tag{5}$$

where the prime, ', indicates matrix transpose. Then substituting $A = \boldsymbol{\Omega}$ in Eq. (2) leads to a simplification:

$$\mathbf{R} = \boldsymbol{\Omega\Omega'} \; . \tag{6}$$

Eq. (6) is a shortcut to the numerical calculation of **R**, but it is something more. It can also be referred to as the unity operator, 𝟙, and written in alternate forms:

$$\mathbb{1} = \Omega\Omega' = \begin{bmatrix} |\omega_1\rangle & |\omega_2\rangle & |\omega_3\rangle \end{bmatrix} \begin{bmatrix} \langle\omega_1| \\ \langle\omega_2| \\ \langle\omega_3| \end{bmatrix} = \sum_{j=1}^{3} |\omega_j\rangle\langle\omega_j| \quad . \tag{7}$$

These factored-out forms of the unity operator can aid in deriving useful formulas. Notice in Eq. (5) and (6) that $\boldsymbol{\Omega'\Omega}$ is a small identity matrix, but $\boldsymbol{\Omega\Omega'}$ is something quite different, projection matrix **R**. With the unity operator, one can write L* as a 3-term series:

$$|L^*\rangle = |\omega_1\rangle\langle\omega_1|L\rangle + |\omega_2\rangle\langle\omega_2|L\rangle + |\omega_3\rangle\langle\omega_3|L\rangle \quad . \tag{8}$$

The coefficients in this “orthonormal function expansion” are the tristimulus values of the light. Squaring both sides shows that the length of the tristimulus vector equals that of L^*. For practical purposes, the tristimulus vector $\boldsymbol{\Omega}'L$ is the same vector as L^*, which ties the orthonormal basis closely to Jozef Cohen’s work. L^* has 471 elements, but its representation in 3-space is the same as $\boldsymbol{\Omega}'L$.