**Question:** How did Jozef Cohen derive the formula $R = A(A'A)^{-1}A'$?

### Cohen’s Derivation of Matrix $R$

Consider a set of $K$ linearly independent column vectors $q_i$ of length $M$. In color work, one might have $K = 3$ and $M = 471$, or $M = 31$, for example. The vectors could be color matching functions or something else. It is assumed in any event that $K < M$. The vectors $q_i$ are the columns of matrix $A_{M \times K}$. If $K = 3$,

$$A = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \vdots & \vdots & \vdots \end{bmatrix}. \quad (1)$$

Then let $N$ be an arbitrary vector of length $M$. In color work, it might represent the spectral distribution of a light. It is then desired to find a vector $N^*$ which is a linear combination of the columns of $A$, and a least-squares best fit to $N$. [In other words, if the columns of $A$ are color matching functions, $N^*$ is a fundamental metamer. The asterisk has nothing to do with complex conjugate or Hermitian. Matrix transpose will be indicated below by the prime symbol, '.]

For illustration, let $K=3$, but the derivation will apply for any $K<M$. Then the columns of $A$ are linearly combined according to the coefficients $U$:

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \quad (2)$$

That is,

$$N^* = AU. \quad (3)$$

Eq. (3) expresses the idea that $N^*$ is a linear combination of the vectors $q_i$. Then the least-squares condition must be applied. Expressing Eq. (3) in more detail,

$$\begin{bmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \\ a_{31}u_1 + a_{32}u_2 + a_{33}u_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} n^*_1 \\ n^*_2 \\ n^*_3 \end{bmatrix}. \quad (4)$$

To minimize the vector distance from $N$ to $N^*$, $|N-AU|$ must be minimized, which is to say one must minimize the following sum of squares, SS:

$$SS = (n_1 - [a_{11}u_1 + a_{12}u_2 + a_{13}u_3])^2 +$$
\[
\begin{align*}
(n_1 - [a_{21}u_1 + a_{22}u_2 + a_{23}u_3])^2 + \\
(n_1 - [a_{31}u_1 + a_{32}u_2 + a_{33}u_3])^2 + \\
(n_1 - [a_{41}u_1 + a_{42}u_2 + a_{43}u_3])^2 + \\
&\ldots
\end{align*}
\]

Now define \( D = N - AU \), so that \( d_j = n_j - [a_{j1}u_1 + a_{j2}u_2 + a_{j3}u_3] \).

With this substitution,

\[
SS = d_1^2 + d_2^2 + d_3^2 + d_4^2 + \ldots
\]  

(6)

Taking the partial derivatives with respect to \( u_1 \), \( u_2 \), and \( u_3 \) and setting them equal to zero gives

\[
\begin{align*}
\frac{\partial SS}{\partial u_1} &= -2a_{11}d_1 - 2a_{21}d_2 - 2a_{31}d_3 - \ldots = 0 \\
\frac{\partial SS}{\partial u_2} &= -2a_{12}d_1 - 2a_{22}d_2 - 2a_{32}d_3 - \ldots = 0 \\
\frac{\partial SS}{\partial u_3} &= -2a_{13}d_1 - 2a_{23}d_2 - 2a_{33}d_3 - \ldots = 0
\end{align*}
\]  

(7)

Eq. (7) amounts to a 3-element vector set equal to zero. Dividing through by the factor of \( -2 \) and then expressing the result as a matrix product,

\[
A'D = 0
\]  

(8)

However, \( D = N - AU \). Thus

\[
A'(N - AU) = 0
\]  

(9)

\[
A'N - A'AU = 0
\]  

(10)

\[
A'AU = A'N
\]  

(11)

\[
U = (A'A)^{-1}A'N
\]  

(12)

We sought an expression for \( N^* \), which equals \( AU \). Therefore,

\[
N^* = AU = A(A'A)^{-1}A'N
\]  

(13)

In Eq. (13), we see that \( A(A'A)^{-1}A' \) is the orthogonal projector projecting \( N \) into the three-space of the color mixture functions, or other column vectors in \( A \). In short,

\[
R = A(A'A)^{-1}A'
\]  

(14)

Eq. (14) is the desired result.


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