**Question:** Why is “Matrix R” important?

**Significance of Matrix R**

**Review:** Consider a spectral power distribution $N(\lambda)$ as a stimulus to vision. A set of color matching functions (CMFs) form the columns of a matrix $A$.

$$
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
\vdots & \vdots & \vdots
\end{bmatrix}.
$$

(1)

Then the fundamental metamer $N^*$ is the linear combination of the CMFs that is a least-squares best fit to $N$. Seeking a means to calculate $N^*$, Jozef Cohen found that $N^* = RN$ when $R$ is given by

$$
R = A(A'A)^{-1}A'.
$$

(2)

Cohen derived Eq. (2) without knowing much of the mathematical context. At first he was pleased just to have an answer, a formula for the fundamental metamer. Later he learned that:

1. Matrix $R$ of Eq. (2) is known to mathematicians as a projection matrix.
2. To find the fundamental metamer, the specific linear combination of CMFs, other methods exist. The Gram-Schmidt process can be applied to the columns of $A$, creating an orthonormal basis. It is then rather easy to derive $N^*$ in terms of the orthonormal functions. A person familiar with orthonormal functions may consider it easier to use them. [Depending on how the orthonormal basis is used, it may use less computer memory, but in 2007 that is no longer important.]

So Cohen at first saw projection matrix $R$ as a solution to one problem, finding the fundamental metamer of a light. Later, he saw that other methods could be used for that problem. But that is not the end of the story.

**Locus of Unit Monochromats.** It is easily proved that $R$ is invariant with respect to the basis used in $A$. (Please see [http://www.jimworthey.com/qna/invariance_of_r.html](http://www.jimworthey.com/qna/invariance_of_r.html).) For instance, if we choose to work with the CIE’s 2° observer, it can be expressed as $x$, $y$, $z$, but also as cone functions, as an orthonormal basis, or as any other set of linear combinations of the initial functions. (The linear transformation must be reversible, involving a matrix whose inverse exists.) Then $R$ is invariant in the strongest possible sense; it is the same big array of numbers in all cases. Since $R$ is closely related to the locus of unit monochromats (LUM), the invariance of $R$ implies that the LUM is an invariant shape for each observer. A different LUM will apply for the 10° observer, and a very different one may apply if the “observer” is a color camera, but in each case the LUM is an invariant embodiment of color matching for that observer.

Suppose that one begins by avoiding Matrix $R$, orthonormalizing the color matching functions, and then generating the LUM as a so-called parametric plot. That is, a narrow-band light of unit power and wavelength $\lambda$ plots to a point in three-space $[\omega_1(\lambda), \omega_2(\lambda), \omega_3(\lambda)]$. Varying $\lambda$ through the spectrum generates the LUM. However the orthonormal set is not unique. Others can be generated to embody the same observer. The invariance of $R$ makes it clear that the shape of the LUM is invariant. Other brief algebra shows that the orthonormal basis generates the same LUM as $R$ would create. Choosing a different orthonormal basis sets different axes, but does not alter the LUM’s shape (except for a possible mirror inversion). Cohen liked to work directly from matrix $R$ and to envision the LUM as floating in space without preferred axes; that approach emphasizes the invariant shape.

So, in short, Matrix $R$ can serve as an important logical link in showing that the LUM is an invariant shape. Matrix
R can be part of the conceptual framework, even if the LUM is drawn by a different method.

**Curve Fitting.** Going in another direction, the projection matrix $R$ is convenient for discussing practical calculations that may arise with color. A problem may be one that we think of as projecting functions into a subspace, but it may also be one that we think of as curve-fitting. If we have a function $\phi$ and we seek $\phi^*$, which is the best fit to $\phi$ by a linear combination of any set of vectors $A$, then it is not necessary to set up and solve the least-squares problem in a series of steps. Just re-word the problem to say that $\phi$ is to be projected into the subspace of vectors $A$. Then compute the matrix $R$ for the particular $A$, and write $\phi^* = R \phi$. The concept of projecting $\phi$ into the column space of $A$ is simple, and so is the computer calculation. In a matrix-oriented computer language, the formula for $R$ can be expressed as one or two lines of code, or as a short function.

**Curve Fitting Example.** Word for word, here is a curve fitting example taken from the poster for CIC 14, the Color Imaging Conference in 2006:

“The more evolved procedure can be called ‘the fit first method.’ The computer code looks like this:

```plaintext
Rcam = RCohen(rgbSens)
CamTemp = Rcam*OrthoBasis
GramSchmidt(CamTemp, CamOmega)
```

Here rgbSens is a matrix whose columns are the 3 camera sensor functions. Rcam is Cohen’s projection matrix $R$ based on the camera functions. OrthoBasis is $\Omega$, the 3 orthonormal vectors for human. CamTemp is then the best fit to OrthoBasis using a linear combination of the camera sensitivities. The columns of CamTemp may not be orthonormal, so Gram-Schmidt finds the orthonormal basis, CamOmega. That’s the main result, and the camera’s LUM is a parametric plot of the 3 columns of CamOmega.”

The second line of the computer code does three curve fits! The short subroutines RCohen() and GramSchmidt() are available at [http://www.jimworthey.com/omatrixcode.html](http://www.jimworthey.com/omatrixcode.html).

**Using matrix $R$ in combination with the orthonormal basis.** As explained at length elsewhere,
1. A set of orthonormal opponent color matching functions, $\Omega$, has many interesting applications.
2. If an orthonormal basis $\Omega$ has been calculated, then the formula for $R$ simplifies: $R = \Omega \Omega'$, where the prime symbol indicates matrix transpose.
3. While the formula $R = \Omega \Omega'$ can be used to compute the large numerical Matrix $R$, it can also be used in brief derivations of useful formulas. For example, one might want to convert a tristimulus vector in the $\Omega$ system to a tristimulus vector in the XYZ system. The needed conversion matrix can be derived by starting with $R = \Omega \Omega'$.

**Conclusions:** When $\Omega$ is in use, $R$ is not needed to generate color vectors, such as those that trace the locus of unit monochromats, but $R$ may well be used for other best-fit problems that arise. The rows of $\Omega$ trace out the LUM, but the invariance of $R$ reassures us that the LUM is an invariant shape.

In other words, the orthonormal basis and vectorial color can at times be used with little reference to matrix $R$. At other times, the projection matrix aids calculation or gives insight.

JAW